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THE STRUCTURE OF VALUATION RINGS

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Dedicated to Molly Sullivan and the memory of Emmy Noether

Introduction**

This paper is a study of valuations of a commutative ring, and the associated valuation rings as defined by Manis [16]. In the classical theory, the valuation v is defined on a field Q, and the valuation ring Q_v is an integral domain with linearly ordered lattice of ideals. Any ring R with the latter property is called a *chain ring*, and any chain ring R is a valuation ring for (a valuation of) its ring of quotients $Q_c(R)$.

In the modern theory, dating from Manis' paper, the valuation ring Q_v is not in general a chain ring, and hence the ideal and ring structure, and homological properties of Q not only become of interest, but are important tools for understanding and classifying valuation rings.

In particular, we show that Manis valuation rings have surprising connections with, and applications to, FPF Ring Theory which arose from non-commutative ring theory as simultaneous generalizations of Nakayama's quasi-Frobenius (QF) rings, Azumaya's pseudo-Frobenius (PF) rings, and, in commutative ring theory, of Prüfer rings and self-injective rings. (See [4,24a] and [32] for the non-commutative, and [5], [6], and [24b] for the commutative FPF rings.)

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** Because of editorial considerations, this introduction was written to replace the table of contents that originally served as introduction. This necessitated certain repetitions since concepts in the text, defined *ab ovo*, could not be deleted. Since the author was unaware that the table of contents was removed until he received galley proofs, it was not possible to rewrite the text to avoid this regrettable redundancy. However, the reader *can* avoid it by skipping the introduction! (There is consolation in that.)

In the classical theory of valuation rings, Krull showed in [14] that every integrally closed domain R is the intersection of valuation rings of its quotient field K of a very special kind. We call a subring V of any commutative ring Q a *conch subring* provided that there exists a unit x of Q not in R with inverse x^{-1} in R, and such that R is maximal with respect to the property of excluding x and including x^{-1}. (The name is after the beautiful seashell that also excludes/includes.) In this case we say that *A conches x in Q*. Now Krull proved that *every integrally closed domain R is the intersection of the conch subrings of $K=Q_c(R)$ that contain R, and, moreover, that every conch subring V of a field K is a chain ring, hence valuation ring, and that either x or x^{-1} lies in V for any nonzero x of K.* (*Loc. cit.*, p. 110, esp. Kriterium 1 u. 2, and Fundamentalsatz der Hauptordnungen, p. 111.) The latter also holds good for any unit x and Manis valuation ring V of any ring Q.

A *pair* of a ring Q is an ordered pair (A,P) consisting of a subring A, and a prime ideal P of A. A *max pair* is a pair maximal in the ordered set of all pairs of Q. (See Section 3.) By Kriterium 2 of [14], if (A,P) is a max pair for a field K, then A is a valuation ring, and P is the unique maximal ideal of A. Manis' theorem is an exact generalization, although A is no longer necessarily a chain ring, or even a local ring, and P is not longer a maximal ideal. (See Section 3.)

As in the classical theory, if A conches x in a ring A, then $(A,\sqrt{x^{-1}A})$ is a max pair of Q. Furthermore, as noted in Section 6, a max pair (A,P) comes from a conch subring A iff $P=\sqrt{x^{-1}A}$ for some unit x of A. Thus, *not every valuation subring is conch.* Moreover, applying the Principal Ideal Theorem for Noetherian Rings, when A is Noetherian, one sees that A must have dimension 1 if conch. Furthermore, any Noetherian chain ring A is a PIR hence must be conch in $Q_c(A)$.

Briefly, a ring is (*right*) *FPF* if all finitely generated faithful right modules generate the category of all (right) modules. In [5,6] the author characterized commutative RPF rings by the two properties: (FPF 1) R has self-injective quotient ring $Q_c(F)$; (FPF 2) every finitely generated faithful ideal of R is a generator; equivalently projective. (See Section 8.)

In Section 12 we prove that *any local FPF ring R is a valuation ring for its quotient ring.* Since $Q_c(R)$ is self-injective for any commutative FPF ring R, then a result proved in Appendix B provides a converse: *A local valuation ring R for self-injective quotient ring is an FPF ring.*

Also in Appendix B we note that *local valuation rings of chain rings are chain rings.* While chain domains are FPF, chain rings in general are not. The property that chain rings enjoy is that every finitely presented faithful module is a generator. Rings with the latter property are called FP^{2}F rings; *and chain rings are characterized among local rings by the property that every factor ring is FP^{2}F* [5]. Moreover, *almost maximal valuation rings are characterized in* [5] *among local rings by the property that every factor ring is FPF.*

Almost maximal valuation rings are fundamental in the classification of FGC rings, or rings over which every finitely generated module decomposes into a direct sum of cyclic modules, and connections with Vamos' fractionally self-injective (FSI) rings, FGC, and FPF rings are pointed out in Section 14.

It is appropriate at this point to cite the characterization of Cunningham [31] of almost maximal valuation rings among chain rings. The quotient sheaf $Q(A_V)$ of a sheaf A of rings extending a ring V is constructible analogously to the way Utumi constructed the maximal quotient ring of a ring, and *Cunningham showed that a chain ring V is an almost maximal valuation ring iff* $Q(A_V)=A_V$. (Note, the underlying topological space is defined by the set of divisorial ideals rather than the set of prime ideals.)

Using conch rings, and FPF rings, we sharpen theorems of Griffin [11,12] on integrally closed subrings of regular rings, and of rings with few zero divisors, by showing these are first of all the intersections of conch subrings, and in the case Q is self-injective, intersections of FPF conch subrings. (See Section 8, and Appendix B.)

The theorem of Eggert [3], also taken up in Section 8, *characterizes when every overring of R in* $Q_{\max}(R)$ *is integrally closed in* $Q_{\max}(R)$ *as Prüfer ring in the sense of Griffin* [11], defined by G4 in Section 7, who characterized them by the same property on overrings of R in the classical quotient ring $Q_c(R)$. When $Q_c(R)$ is von Neumann regular, then Griffin [12] characterized a Prüfer ring as being semihereditary (see Proposition G5 in Section 7), a theorem which applies to Eggert's theorem when R is semiprime, and in this case $Q_c(R)=Q_{\max}(R)$ (see Eggert's theorem, Section 8). We show in Section 8 for these rings that R is Prüfer iff FPF. More generally, if $Q_c(R)$ is injective, then R is Prüfer iff FPF (Section 8, Proposition).

In Sections 11 and 12 we study the uniqueness of the associated valuation ring W of a valuation ring $A=Q_v$ of a ring Q. If H is the ideal at ∞, then H is a prime ideal of Q contained in A, and W is the valuation ring of the valuation v_K of the quotient field $K=Q_c(Q')$, where $Q'=Q/H$, extending the valuation v' of Q' induced by v. (It can be shown that K is also the quotient field of $A'=A/H$, but this is left for a sequel.)

If A is a maximal subring of Q, say $Q=A[x]$, *then A conches x in Q, and W is a rank* 1 *valuation ring for K. If, in addition, W is a discrete valuation ring, then W is an n-radical extension of the local ring* $A'_{P'}$ *of* A' *at* P' *in the sense that* $w^n\in A'_{P'}$ *for all* $w\in W$. *Moreover, if M is the radical of W, then* $M^n=x'^{-1}W$, so $M=\sqrt{x'^{-1}}W$. (Thus, by the Conch Ring Theorem, W conches x' in K.)

This structure theorem allows us to apply the ubiquitous theorem of Kaplansky [13a] on the structure of radical extensions of fields to deduce that *A has at most one associated discrete valuation ring, namely* $A'_{P'}$, *when A is a conch maximal subring of Q with residue ring A/P of characteristic* 0. (In effect, Kaplansky's theorem implies that $n=1$ in this case.)

1. Valuations

In [16] Manis extended classical valuation theory for integral domains to arbitrary commutative rings with unit: a *valuation* υ of a ring Q is a mapping

$$\upsilon : Q \to \Gamma \cup \{\infty\}$$

where Γ is a totally ordered abelian group (and ∞ is the symbol with the usual properties of ∞!) such that

$$\upsilon(ab) = \upsilon(a) + \upsilon(b),$$

$$\upsilon(a+b) \geq \min\{\upsilon(a), \upsilon(b)\}$$

holds for all $a, b \in Q$. The *valuation ring of* υ is the subring

$$Q_\upsilon = \{a \in Q \mid \upsilon(a) \geq 0\}$$

and

$$P_\upsilon = \{a \in Q \mid \upsilon(a) > 0\}$$

is the *valuation prime ideal* of υ. If $\Gamma \neq 0$, then υ is said to be *proper* (otherwise trivial) and genuine if $P_\upsilon \neq Q_\upsilon$ and $P_\upsilon \neq 0$.

2. Chain rings

A ring A is called a *chain ring* provided that the lattice of ideals is linearly ordered. If A is an integral domain and a chain ring, then A is called a *chain domain*. A necessary and sufficient condition for a domain A to be a chain domain is that A be the valuation ring of a valuation υ of its quotient field $Q_c(A)$. Every chain ring is a valuation ring but not conversely.

3. Max pairs

A *pair* for a ring Q is an ordered pair (B, L) where B is a subring of Q and L is a prime ideal of B. The set pairs Q of all pairs of Q is ordered by the relation $(B, L) \supseteq (C, M)$ iff $B \supseteq C$ and $L \cap C = M$. The set of all pairs of Q is an inductive st, and so is the set $\text{Pairs}_Q(B, L)$ of all pairs (A, P) containing a given pair (B, L). By Zorn's lemma $\text{Pairs}_Q(B, L)$ contains at least one maximal pair (A, P), and we call any such a *max pair of* Q. It is obvious that (A, P) is a max pair iff (A, P) is maximal in $\text{Pairs}_Q(A, P)$. A theorem of Manis [16] characterizes the valuation subrings of a ring Q by the max pairs of Q.

Manis' Theorem. *If υ is a valuation of Q, then (Q_υ, P_υ) is a max pair of Q; conversely, if (A, P) is a max pair, then there exists a valuation υ of Q for which $A = Q_\upsilon$ and $P = P_\upsilon$.*

From classical valuation theory we know that if Q is a field, then (A, P) is a max pair iff A is a chain ring and $P = \text{Max}\, A$ is the unique maximal ideal ([14], [15], [22]).

Moreover, in this case necessarily $Q_c(A)=Q$ since $q^{-1}\in A$ whenever $q\in Q\backslash A$.

A prime ideal M of Q defines a max pair (Q,M) of Q, so in general one cannot say much more ring theoretical about a valuation subring A of Q than about Q itself. A typical theorem is that A is semihereditary if Q is von Neumann regular (Griffin [11]).

4. The core of a valuation

The *core of* a valuation υ of Q

$$H=H_\upsilon=\{a\in Q \mid \upsilon(a)=\infty\}$$

is a prime ideal of Q, and is the maximal ideal of Q contained in Q_υ. Let Q' denote the residue ring Q/H, and let $K=Q_c(Q')$ be its quotient field. The valuation υ induces a valuation $\upsilon_K: K\to\Gamma\cup\{\infty\}$ sending $a'(b')^{-1}$ onto $\upsilon(a)-\upsilon(b)$ for all $a\in Q$ and all $b\in Q\backslash H$. This valuation υ_K is called the *associated valuation*, and K the *associated field* of υ. The *associated valuation ring*

$$K_{\upsilon_K}=\{k\in K \mid \upsilon_K(k)\geq 0\}$$

is thus a valuation domain, hence a chain ring, that contracts to Q'_υ:

$$K_{\upsilon_K}\cap Q'=Q'_\upsilon;$$

and its maximal ideal P_{υ_K} contracts to P'_υ:

$$P_{\upsilon_K}\cap Q'=P'_\upsilon.$$

Thus, while neither Q_υ nor Q'_υ are chain rings in general, Q'_υ is the contraction (to Q') of a chain ring K_{υ_K} of the associated field.

5. The quotient ring of a valuation ring

Let Q^* denote the set regular elements of a ring Q, and

$$Q_c(Q)=Q_{cl}(Q)=\{ab^{-1} \mid a\in Q, b\in Q^*\}$$

denote its full or classical quotient ring. While any valuation υ of Q has a unique extension to $Q_c(A)$, in general the valuation theory can be carried out without assuming that $Q=Q_c(Q)$, and we propose to do so. For example, if R is a subring of Q, and (R,L) is a pair, then possibly the max pair $(A,P)\supseteq(R,L)$ may have $Q_c(A)=Q$ even if $Q_c(R)\neq Q$. Even if $Q_c(A)\neq Q$, we are able to show:

Theorem. (1) *The associated field of a valuation ring A of Q is the quotient field of A': $Q_c(A')=Q_c(Q')$.*

(2) *A is an essential A-submodule of Q, hence $Q_c(A)\subseteq Q_c(Q)$.*

(3) *If Q is generated by A and units (e.g. if Q is a local ring), then* $Q_c(A) = Q_c(Q)$.

Corollary. *If Q is von Neumann regular ring, then* $Q = Q_c(A)$.

Moreover, one can characterize ring-theoretically the valuation s brings of von Neumann regular rings.

Theorem. *If Q is a von Neumann regular ring, then a subring A is a valuation subring of Q iff A contains a maximal ideal H of Q such that $A' = A/H$ is a chain domain with $Q_c(A') = Q' = Q/H$. When this is so, then $Q_c(A) = Q$, and H is the core of the valuation.*

6. Conch rings

Let A be a maximal order of a ring Q. Then there is an element $y \in A$ with $x = y^{-1} \notin A$, and A is a subring that is maximal with respect to containing x^{-1} and excluding x. We next consider a wider class of valuation rings, called conch rings, having this property.

A *conch* subring of Q is a subring V maximal with respect to excluding a given unit x of Q and including x^{-1}. Then V will be said to *conch* x in Q, and will be called an *x-conch subring* of Q.

Conch subrings were first introduced by Krull in his monograph "Idealtheorie" [14], where he showed that any integrally closed subring R of a field K is the intersection of the conch subrings of K containing R, and that every conch subring of K is a chain ring. We generalize this as follows:

Theorem. *Let Q be a commutative ring.*

(1) *Every conch subring A of Q is integrally closed.*

(2) $(A, \max A)$ *is a max pair of Q.*

Corollary. *Every conch subring of Q is a valuation ring.*

The converse does not hold however:

Conch Ring Theorem. *If (A, P) is a max pair of Q, then A is a conch subring iff $P = \sqrt{x^{-1}A}$ for some $x \in U(Q)$, where $U(Q)$ is the units group.*

Thus, not every valuation subring A of Q is a conch subring. Nevertheless, when there are enough units, the t classes of rings have the same intersection; to wit, Krull's theorem for fields alre cited. We extend this to any integrally closed subring R of a von Neumann regu ring: $R = \mathrm{Conch}_Q(R)$. (Compare the theorem of Griffin [11] discussed presently.)

Theorem. *If R is a subring of Q, then the intersection* $\mathrm{Conch}_Q(R)$ *of the conch subrings of Q containing R is integrally closed. Moreover, every element of*

$$\mathrm{Conch}_Q^*(R) = \mathrm{Conch}_Q(R) \cap U(Q)$$

is integral over R.

We call $\mathrm{Conch}_Q(R)$ the $*$-integral closure of R.

We say that the *Conch Intersection Theorem* holds for Q, if every integrally closed subring $R = \mathrm{Conch}_Q(R)$.

7. Griffin's theorems

A ring R has *few zero divisors* (in Griffin's sense) if it has just finitely many maximal prime ideals of 0. (Equivalently $Q = Q_c(R)$ is *semilocal*, i.e. $Q/(\mathrm{rad}\, Q)$ is semisimple.) The principal property of these rings used in Griffin's theorems is this: if $z \in Q_c(R)$ and if $a \in R^*$, there exists $u \in R$ so that $z + au \in Q^*$. ("Choose u in all max 0-prime ideals not containing z and in no 0-prime ideal containing z.")

Theorem (Griffin [11]). *If Q is von Neumann regular, or has few zero divisors, and if R is integrally closed in $Q = Q_c(R)$, then R is the intersection of valuation subrings of Q.*

To introduce further theorems we need additional concepts, especially those of the large quotient ring $R_{[S]}$ of R with respect to a *multiplicatively closed subset* (=ms.) S of R. Let $S^* = R^* \cap S$.

(Q1) R_S denotes the ring of quotients w.r.t. S.

(Q2) $R_{(S)} = R(S^*)^{-1} = \{rs^{-1} \in Q_c(R) \mid r \in R, s \in S^*\}$.

(Q3) $R_{[S]} = \{x \in Q_c(R)$ (resp. $x \in Q_m(R) \mid \exists s \in S$ so that $xs \in R\}$.

$R_{(S)}$ is called the *quotient ring of R w.r.t. S*. $R_{[S]}$ is called the *large quotient ring* of R in $Q_c(R)$ (resp. in $Q_m(R)$) w.r.t. S.

If M is a prime ideal, and $S = R \setminus M$, then we set $R_{(P)} = R_{(S)}$ and $R_{[M]} = R_{[S]}$.

G1. Proposition (Griffin [11]). *If R is integrally closed in $Q_c(R)$, then so is $R_{[S]}$ for any multiplicative closed set S.*

G2. Proposition (Griffin [11]). *If R is a ring with few zero divisors, then $R_{[P]} = R_{(P)}$ for every prime ideal P.*

An ideal I of R is *regular* if $I \cap R^* \neq \emptyset$.

The *regular total order property* at a prime ideal P is the property that if I and K are ideals of R one of which is regular, then either $IR_P \supseteq KR_P$ or $KR_P \supseteq IR_P$. Then R_P is said to have the *regular total order property*. Trivially, if R_P is a chain ring, this holds.

The *core* of a prime ideal P of R is defined as

$$\mathrm{Core}_R\, P = \{b \in R \mid \forall r \in R^* \ \exists e \in R \setminus P \ni ber^{-1} \in R\}.$$

If $P = P_\upsilon$ for a valuation υ of Q, the core of P is simply the core of υ.

G3. Proposition (Griffin [11]). *Let M be a maximal ideal of R. Then the following are equivalent.*

(M1) *$R_{[M]}$ is a valuation ring for $Q_c(R)$.*

(M2) *If $a, b \in R$, and $b \notin \mathrm{core}_R P$, then $\exists s \in R$, $t \in R \setminus P$ so that $sa = tb$.*

(M3) *If I, K are ideals of R not both contained in the core of P, then $IR_P \supseteq KR_P$ or $KR_P \supseteq IR_P$.*

(M4) *R_P has the regular total order property.*

G4. Proposition (Griffin [11]). *The following are equivalent for a ring R:*

(P1) *$R_{[M]}$ is a valuation ring for $Q_c(R)$ for every maximal ideal M of R.*

(P2) *R_M has regular total order property for every maximal ideal M.*

(P3) *Every overring of R in $Q_c(R)$ is flat over R.*

(P4) *Every overring is integrally closed in $Q_c(R)$.*

(P5) *Every finitely generated regular ideal is invertible in $Q_c(R)$.*

(P6) *Every regular ideal generated by two elements is invertible in $Q_c(R)$.*

A ring R is *Prüfer* if it satisfies any one (hence all) of the properties (P1)–(P6).

G5. Proposition (Griffin [11, 12]). *The following are equivalent conditions on a ring R:*

(1) *R is semihereditary.*

(2) *R is Prüfer and $Q_c(R)$ is regular.*

(3) *$Q_c(R)$ is zero-dimensional and $R_{[M]}$ is a chain domain for every maximal ideal M.*

G6. Corollary (Griffin). *Every integrally closed subring R of a von Neumann regular quotient ring $Q = Q_c(R)$ is an intersection of semihereditary valuation subrings.*

8. FPF rings

We next relate conch subrings to FPF rings. A ring A is *FPF* iff every finitely generated faithful module M generates mod-A, equivalently $M^n \approx A \oplus X$ in mod-A for some $n \geq 1$. The chief examples of FPF rings are Prüfer domains, hence Dedekind rings, self-injective rings, and their arbitrary direct products [5, 6]. Furthermore FPF rings are characterized in [6] by the two properties:

(FPF 1) $Q_c(A)$ is injective.

(FPF 2) Every finitely generated faithful (=dense) ideal of A generates mod-A; equivalently is projective.

The condition (FPF 2) is also called *pre-FPF*.

Semiprime FPF Theorem [5]. *A semiprime ring A is FPF iff A is a semihereditary* (FPF 1) *ring.*

Any FPF ring A is integrally closed in its quotient ring $Q_c(A)$ ([6], p. 78, Proposition 2.7); hence the intersection $R=\bigcup_{i\in I}A_i$ of an arbitrary family $\{A_i\}_{i\in I}$ of FPF subrings of a ring Q with each $Q_c(A_i)=Q$ is integrally closed in Q.

We say that the *FPF Intersection Theorem* holds for a (necessarily self-injective) ring Q when every integrally closed subring R of Q is such an intersection. Moreover, if the $\{A_i\}_{i\in I}$ can be chosen to be FPF conch subrings, we say the *FPF Conch Intersection Theorem holds for Q.*

Theorem. *A valuation ring A of a self-injective von Neumann regular ring Q is FPF, and, moreover, the FPF Conch Intersection Theorem holds for Q.*

If $Q_c(A)=Q$, then by Griffin's theorem, A is semihereditary hence pre-FPF. Since Q is self-injective, A is then FPF iff $Q_c(A)=Q$, a result which follows from our results on quotient rings of valuation rings described earlier. If R is any subring of a regular of Q containing all idempotents of Q, and if S is any subring $\supset R$, then S contains a unit of Q not in R: if $s\in S\setminus R$, and $e=e^2\in Q$ generates sQ, then $x=s+(1-e)$ is the desired unit. Thus, in view of our result that $\mathrm{Conch}_Q(R)$ is the $*$-integral closure of R in Q, we conclude that $R=\mathrm{Conch}_Q(R)$ when R is integrally closed.

Corollary. *If R is nonsingular and integrally closed in $Q=Q_{\max}(R)$, then R is the intersection of the FPF conch subrings of Q containing R.*

This follows since Q is a self-injective regular ring.

Note that this shows that in the theorem Q need not be $Q_c(R)$ as in Griffin's theorem, and furthermore, Q need not be $Q_{\max}(R)$.

Another aspect of the FPF Theorem is that any overring of an FPF ring A is also FPF, hence integrally closed in $Q_c(A)$. This fact recalls the study of Eggert [3] of a ring R with the property, called I-ring, that every overring S of R in $Q=Q_{\max}(R)$ is integrally closed.

Theorem (Eggert [3]). *An I-ring R is a Prüfer ring. Moreover, if R is semiprime,* $Q=Q_{\max}(R)=Q_c(R)$.

Thus, a semiprime I-ring has injective $Q_c(R)$. This is the point of departure for the next theorem.

Proposition. *The following are equivalent conditions on a ring R with injective $Q_c(R)$.*

(1) *R is pre-FPF.*

(2) *R is Prüfer.*

(3) *Every overring of R in $Q_c(R)$ is integrally closed.*

(4) *Every overring of R in $Q_c(R)$ is flat over R.*

(5) *R is an I-ring.*

When this is so, then R is FPF.

The proof makes use of the next result.

Lemma. *If $Q = Q_c(R)$ is injective, then a finitely generated ideal I of R is faithful iff I is regular.*

Obviously, if I is regular, it is faithful, and conversely, if Q is injective, then I faithful implies by a theorem of Nakayama and Ikeda (in [4], Chapter 23) that $IQ = Q$. Write

$$1 = \sum_{i=1}^{n} x_i q_i$$

in Q, where $x_i \in I$, $q_i = r_i c^{-1} \in Q, c, r_i \in R$, $i = 1, \ldots, n$. Then $c = \sum_{i=1}^{n} x_i r_i \in I$, so I contains a regular element.

This shows the equivalence (1)$\Leftrightarrow$(2) and the other equivalences are derived from Griffin's theorems.

Corollary. *The following are equivalent conditions on a ring A.*

(1) *A is FPF.*

(2) *A is an I-ring with injective $Q_c(A)$.*

Corollary. *A maximal order A of a self-injective ring Q is an FPF conch ring.*

We previously remarked that a maximal order A of a ring Q is a conch subring, hence integrally closed in Q, so A is FPF by the corollary.

Corollary. *A semiprime ring R is FPF iff R is an I-ring.*

For using Eggert's result, R has injective $Q_c(R)$.

9. Continuous rings

A ring R is said to be *right continuous* ([20], [21]) if the following two conditions are satisfied:

(C1) If I is a right ideal of R, then there is a maximal essential extension of I in R generated by an idempotent.

(C2) If $f=f^2\in R$, and if $I\approx fR$, then I is generated by an idempotent.

Right continuous rings get their name by upper continuity property of the lattice of principal right ideals [21] (expounded in [10a], p. 160 ff.). A ring R is *continuous* if it is right and left continuous. An integral domain R not a field is an example of a ring satisfying (C1) but not (C2).

We now cite some theorems on continuous rings of Utumi (but see [10a], Chapter 13), and deduce some corollaries for integrally closed rings.

Utumi Theorems ([20, 21]). **1.** *A right self-injective ring R is right continuous.*

2. *If R is right continuous, then R/J is right continuous, and von Neumann regular, and J, the right singular ideal, coincides with the Jacobson radical.*

3. *A regular ring R is right continuous iff R contains all idempotents of its maximal right quotient ring* $Q_{\max}(R)$.

Corollary. *If Q is a self-injective regular ring generated by its idempotents, then Q has no integrally closed regular subrings* $\neq Q$.

Corollary. *If R is a commutative regular ring integrally closed in* $Q=Q_{\max}(R)$, *then R is a continuous ring.*

Let R denote a commutative regular ring. Then R is self-injective iff R is complete (= the lattice of principal ideals is complete). Thus, $Q=Q_{\max}(R)$ is complete.

If R is a Boolean ring, then R is a regular ring $\neq Q=Q_{\max}(R)$ whenever R is not complete. Then R cannot be integrally closed in Q since every element of Q is an idempotent. Similarly, R cannot be continuous.

A theorem of Goodearl [10b] characterizes regular rings which are generated by their idempotents.

Theorem. *A regular ring R is generated by its idempotents iff no division ring except possibly* $\mathbb{Z}/p\mathbb{Z}$ *is a homomorphic image of R.*

Corollary. *If a commutative regular ring R is generated by idempotents, then the only residue fields of R are finite prime fields.*

10. Conch rings as maximal subrings

If A conches x in Q, then A is a maximal subring of Q iff $Q=A[x]$. If Q is a field, this happens if and only if A is a rank 1 valuation ring. In this section we discuss this situation.

If A conches x in Q, then A conches x in $Q_2=A[x]$, and hence A is also a valuation ring of Q_2. Moreover, as stated, A is a maximal subring of Q_2. Since Q_2 is generated by A and units, then $Q_c(A)=Q_c(Q_2)$ by Section 5.

The largest ideal $H_2=H_{Q_2}(A)$ of Q_2 in A is $A_\infty=\bigcap_{n>0}x^{-n}A$. (Clearly $A_\infty \subsetneq H_2$, and if $q\in H_2$, then $qx^n\in A\ \forall n$, so $q\in A_\infty$.) We set $Q_2'=Q_2/H_2$, $A_2'=A/H$, etc.

Theorem. *If A conches x in $Q_2=A[x]$, then any subring of W of $K_2=Q_c(Q_2)$ that conches x' in K_2 and that contains A_2' is a rank-1 valuation domain (equivalently $\bigcap x'^{-n}W=0$) and $W\cap Q_2'=A_2'$.*

First $W\cap Q_2'$ conches x' in Q_2'. Since A_2' also conches x' in Q_2', then $W\cap Q_2'=A_2'$. Next, by Krull's Theorem, W is a chain domain, that is, a classical valuation domain of a field. Since

$$K=Q_c(W)=Q_c(A_2')=Q_c(Q'),$$

we have

$$\bigcap x'^{-n}A_2'=0\Rightarrow\bigcap x'^{-n}W=0.$$

Since W conches x',

$$M=\max W=\sqrt{x'^{-1}W}$$

and if P_2 is any prime ideal $\neq M$, then

$$W_{P_2}=W[x]=K_2$$

proving that $P_2=0$. Thus, W is rank-1.

Henceforth, to simplify notation, we shall assume that A is a maximal subring of $Q=A[x]$. Then, $H_2=H$, $Q_2'=Q'$, $A_2'=A'$, etc. The rest of the section is devoted to the question: when is W of the theorem, which we call the *associated rank 1-valuation domain of A*, a discrete valuation ring? We prove that if it is, then W is a 'radical' extension of the local ring $A'_{P'}$, where a ring R is a *radical extension* of a subring S in case for each $x\in R$ there corresponds an integer $n>0$ such that $x^n\in S$. If there is an integer $n>0$ so that $x^n\in S$ for all $x\in R$, then we say that R is an *n-radical* (or *n-ical*) *extension* of S. (Compare [7].)

Theorem. *Let A be an x-conch maximal subring of Q, and suppose there exists an associated discrete valuation ring W of A. Then W is an n-radical extension of the local ring $A'_{P'}$ of A' at P', where $P=\sqrt{x^{-1}A}$, and*

$$M=\max W=\sqrt{x'^{-1}W},\quad\text{and}\quad M^n=x'^{-1}W.$$

11. Kaplansky's theorem on radical extensions revisited again[1]

Using a theorem of Kaplansky [13] on radical extensions of fields, we conclude

[1] It was revisited in [7].

that if $W/M \neq Q_c(A'/P')$, then W/M has characteristic $p>0$ and either

(KAP 1) W/M is purely inseparable over $Q_c(A'/P')$, or else

(KAP 2) W/M is an algebraic extension of a finite field.

Consequently, we have the:

Corollary. *If A is a conch maximal subring of Q such that A/P has quotient field of characteristic* 0, *then there is at most one associated discrete valuation ring of A, namely*

$$W = A'_{P'}$$

the local ring of A' at P'.

This follows from the theorem since necessarily $W/M = Q_c(A'/P')$, so if $S = A' \setminus P'$, then $W/M = (A'/P')S^{-1}$ and so

$$W = A'S^{-1} = A'_{P'}.$$

Corollary. *If A is an x-conch maximal subring of Q, then there is at most one associated discrete valuation ring W having* max $W = x'^{-1}W$, *namely,* $A'_{P'}$.

For, by the theorem, W is n-radical over $A'_{P'}$ and $n=1$.

12. Local FPF rings are valuation rings

Recall that a *waist* of a ring R is an ideal W such that every ideal of R either contains or is contained in W.

Local FPF Ring Theorem ([5]). *A ring R is a local FPF ring iff R has injective* $Q = Q_c(R)$ *and the set of zero divisors of R is a waist W such that R/W is a chain ring.*

A ring R is a *sandwich ring* if R contains the radical $J = J(Q)$ of $Q = Q_c(R)$.

Sandwich Ring Theorem ([5], [6]). (1) *Any local FPF ring R is a sandwich ring.*

(2) *If R is a sandwich ring, then R is FPF if and only if* $Q = Q_c(R)$ *is injective and R/J is FPF.*

Note. The proof of the theorem involves showing that $Q_c(\bar{R}) = \bar{Q}$, where $\bar{Q} = Q/J$, and $\bar{R} = R/J$. Since $\bar{Q}$ is injective by Utumi's theorem (Section 9) when Q is, then R will be FPF iff (FPF 2) holds for $\bar{R}$.

We next remark how the sufficiency of the Local FPF Ring Theorem follows from (2) of the Sandwich Ring Theorem: if I is any finitely generated faithful ideal of R, then (via injectivity of Q), I contains a regular element, so $I \supset W$. Then using the chain ring property of R/W, one sees that $I = yR + W$ for some $y \in R$, hence $I = yR \approx R$, so (FPF 2) holds in R.

We may now deduce:

Theorem. *Any local FPF ring R is a valuation ring for $Q_c(R)$.*

The proof is almost immediate from the Local FPF Ring Theorem: by injectivity of Q, J consists of zero divisors. Since R is a sandwich ring, $W = J$. It follows that (R, J) is a max pair of Q (inasmuch as $(\bar{R}, \bar{J})$ is one for $\bar{Q}$), so R is a valuation ring.

13. Local valuation rings are sandwich rings

A *local valuation ring A* is a valuation ring that is a local ring.

A *sandwich subring A* of Q is a subring containing the radical $J = \text{rad}\, Q$ of Q.

Local Valuation Ring Theorem. *Any local valuation ring A of Q is a sandwich subring.*

Any valuation ring A has the property that $q \in A$ or $q^{-1} \in A$ for any $q \in U(Q)$, so the next lemma suffices for the proof.

Sandwich Subring Lemma. *Let A be a local subring containing q or q^{-1} for every unit q of Q. Then A is a sandwich subring.*

Let $j \in J \setminus A$ and let $j_0 \in J$ be such that $(1+j)^{-1} = (1+j_0)$. Since $q = 1 + j \in U(Q) \setminus A$, $q^{-1} = 1 + j_0 \in A$. Since $q = (1+j_0)^{-1} \notin A$, $j_0 \notin \max A$, hence $j_0^{-1} \in A$. Thus J contains a unit, a contradiction. This proves that $J \subseteq A$.

Corollary. *If A is a local valuation ring for Q, and if all of the prime ideals of Q containing J are maximal, then $Q' = Q/H$ is a field, and A' is a chain domain for Q'.*

The corollary follows from the theorem in as much as $J \subseteq A$ and this implies that H, the largest ideal of Q contained in A, contains J, so H is a maximal ideal. Now if A is any valuation ring for Q, then A' is one for Q'. However, by the first theorem of Section 5, $Q_c(A') = Q'$, and every valuation ring of a field is a chain domain by the classical theorem (Section 2).

Partial Converse of Local Valuation Ring Theorem. *If A is a valuation sandwich subring for Q, and if $Q' = Q/H$ is a field, then A is a local ring.*

For, since $A' = A/H$ is a valuation ring for Q', A' is a chain domain by the classical theorem stated in Section 2.

Let $M \supseteq H$ be the ideal of A such that

$$M' = M/H = \max A'.$$

Then, if $a \in A \setminus M$, then $a \notin J$, so $b = a^{-1} \in Q$. Since A' is a chain domain, $b' = (a')^{-1} \in A'$, hence $b \in A$, proving that A is a local ring with $\max A = M$.

Corollary. *If A is a valuation sandwich subring for a local ring Q, then A is a local ring.*

For then Q/J is a field, so $J = H$ and $Q' = A/J$. We can sharpen the above results for a conch subring.

Local Conch Ring Theorem. *Let A conch x in Q. Then A is a local ring with* $\max A = \sqrt{x^{-1}A}$ *iff Q is a local ring and A is a sandwich subring.*

First assume that A is local, that $P = \max A = \sqrt{x^{-1}A}$ and let $q \in Q \setminus A$. Then we can write

$$x = a_0 + a_1 q + \cdots + q_n q^n$$

for $a_i \in A$, $i = 0, \ldots, n$, and $n \geq 1$. Then

$$x = a_0 + q_0$$

where

$$q_0 = a_1 q + \cdots + a_n q^n = qb \quad \text{and} \quad b = a_1 + \cdots + a_n q^{n-1},$$

so

$$1 = x^{-1} a_0 + x^{-1} q_0 \quad \text{and} \quad c = x^{-1} q_0 = 1 - x^{-1} a_0 \in A.$$

Since $x^{-1} a_0 \in P = \sqrt{x^{-1}A}$, $c \notin P = \max A$, hence $c^{-1} \in A$. Since $q_0 = xc$, $q_0 = qb$ is a unit of Q and $q_0^{-1} = x^{-1} c^{-1}$, so q is a unit of Q.

We next show that any $a \in A \setminus H$, where $H = H(A)$, is a unit of Q. Since $q = at \notin A$ for some $t \in Q$, q whence a is a unit, as required. This proves that H is a maximal ideal of Q consisting of the non-units of Q, so Q is local. Since $H = \operatorname{rad} Q$, A is a sandwich subring.

Conversely, if Q is a local ring and $J \subseteq A$, then A' is a chain ring for $Q_c(A') = Q'$. Suppose $M \supseteq J$ is the ideal of A such that $M' = \max A' = \sqrt{x'^{-1}A'}$. Clearly, $M = \sqrt{x^{-1}A}$ since $x^{-1}A \supseteq J$. If $a \in A \setminus M$, then $a^{-1} \in Q$, so $a \in A^*$. Now if $a \notin U(A)$, then we can write as before

$$x = a_0 + a_1 q + \cdots + a_n q^n$$

for $q = a^{-1}$, and then

$$x a^n = a_0 a^n + \cdots + a_n \in A,$$

so $a \in M = \sqrt{x^{-1}A}$. This contradiction proves that A is local with $\max A = \sqrt{x^{-1}A}$.

The local part of the converse follows from the last corollary also since any conch subring is a valuation subring (Section 6).

14. Almost maximal valuation rings

In Section 1 we remarked:

Theorem. *Every chain ring A is a valuation ring for $Q = Q_c(A)$.*

Although this follows from the regular total order property and Griffin's Theorem in Section 7 we can prove this directly: A and $P = \max A$ define a max pair in Q, since if $(B, L) \supseteq (A, P)$ and if $b = ac^{-1} \in B \setminus L$, where $a \in A$, $c \in A^*$, then $a \notin P$, so $a^{-1} \in A$, whence $b^{-1} = a^{-1}c \in A$. Then $c \in P$, unless $b \in A$. But then $c^{-1} = ba^{-1} \in B$, hence $1 = c^{-1}c \in BP \subsetneq L$, a contradiction. Thus, $B \setminus L \subseteq A$, and so $B \subseteq A$, that is, $B = A$, whence $(B, L) = (A, P)$ is a max pair.

We say that a chain ring R is a *maximal valuation ring* if every system $x \equiv x_i \pmod{I_i}$ of congruences is solvable for x in R provided that it is finitely solvable for x in the sense that every finite subset of congruences is solvable.

A chain ring R is an *almost maximal valuation ring* (AMVR) provided that R/I is a maximal valuation ring[2] (*MVR*) for every ideal $I \neq 0$.

AMVR's were first introduced by I. Kaplansky [13b] who prove that they had the property (FGC) that *finitely generated modules are direct sums of cyclic modules.* Much later D.T. Gill [25] generalized this:

Theorem (Kaplansky, Gill et al.). *A local ring R is an AMVR iff R is an FGC ring.*

Another characterization:

Matlis' Theorem ([27]). *A chain domain R is an AMVR iff $Q_c(R)/R$ is injective.*

And yet another:

Gill's Theorem ([25]). *A local ring R is an AMVR iff the injective hull $E(R/\max R)$ is a uniserial* (*or chain*) *module.*

The Kaplansky-Gill et al. and Gill Theorems are contained in [4], p. 134, Theorem 20.49.

W. Brandal, T. Shores, R.&.S. Wiegand and P. Vamos completed the structure theory for FGC rings, and this is fully described in Brandal's Lecture Notes [23].

[2] The term is justified in [28]: a MVD is a valuation ring for a maximally complete field: If υ is a valuation of a field K, then the valuation ring P_υ is maximal iff every extension of υ to an overfield enlarges either the value group or the residue field ([28], [29]; also compare [30], [31]).

However, Vamos' characterization [29] is particularly suited for our purposes, so we shall describe it.

A ring R is *fractional self-injective* (FSI) iff every factor ring R/I has injective quotient ring $Q_c(R/I)$. In [5] I conjectured that a ring R is FSI iff every factor ring R/I is FPF. These latter are called CFPF rings, and I proved the conjecture in [6].

Vamos' Theorem ([26]). *A ring R is FGC iff R is FSI and Bezout (=finitely generated ideals are principal).*

Theorem ([6]). *R is FGC iff R is CFPF.*

Corollary ([6]). *A local ring R is CFPF iff R is an AMVR.*

15. Split null extensions as valuation rings

In [24a,b] we studied when the split null extension $R=(B,E)$ of a bimodule E over ring B is (F)PF, or a chain ring, or when R had relatedproperties (e.g. self-injective). For example, Proposition 5A of [24a] states that R is a right chain ring iff B is a right chain ring and E is a uniserial right B-module such that $bE=E$ for every $0\neq b\in B$. *If B* is commutative and E is faithful, then B must be a domain for this. If, further, E is torsionfree, then E must be injective (Corollaries 5B and 5C).

Theorem 6 of [24a] characterizes when $R=(B,E)$ is a PF chain ring for commutative B. This happens iff B is an AMVR domain such that $B=\mathrm{End}_B E$ and E is the injective hull of $B/\max B$. (See [24a] for other characterizations.) *Henceforth, assume that E is faithful over B.*

For commutative B, in [24b] we proved that $R=(B,E)$ is FPF iff E is injective over B, $\mathrm{End}_B E$ is the quotient ring BS^{-1} of B with respect to the multiplicative set S consisting of all $b\in B$ with zero annihilators in E, and every finitely generated ideal which is faithful on E is invertible in BS^{-1}. It is much easier to characterize when $R=(B,E)$ is a valuation ring for $Q_c(R)=(BS^{-1},ES^{-1})$:

Theorem. *Split-null extension $R=(B,E)$ is a valuation ring for $Q_c(T)=(BS^{-1},ES^{-1})$ iff $E=Es$ for every $s\in S$, and B is a valuation ring for BS^{-1}, assuming that E is faithful.*

Corollary. *If E is torsion free over B, this happens iff E is divisible and B is a valuation ring for $Q=Q_c(B)$.*

Corollary. *If E is a torsionfree module over a domain B, then $R=(B,E)$ is a valuation ring for $Q_c(R)$ iff E is injective and B is a chain domain.*

Appendix A

We give a sketch of the proof of the Conch Ring Theorem. For a domain A, we already have indicated that any conch subring A of a field K is a chain domain. The statement that A conches x in K is simply that $A[x]$ is the unique minimal subring of K (properly) containing A. In view of the 1-1 correspondence $M \mapsto A_M$ between prime ideals M of A and over-rings of A, it follows that $P = \operatorname{rad} A$ contains a largest ideal $P_2 \neq P$. Then, using the fact that A conches x, we can easily show that $\sqrt{x^{-1}A}$ is a prime ideal, whence $P = \sqrt{x^{-1}A}$ (since $x^{-1} \notin P_2$).

Conversely, if a chain domain A has radical $P = \sqrt{x^{-1}A}$, then one can show that $P_2 = \bigcap_{n \geq 1} x^{-n}A$ is a prime ideal, and the largest prime ideal contained in P, hence $A[x] = A_{P_2}$ is the intersection of all subrings of K properly containing A. Thus, A conches x.

The proof for the general case of a valuation ring A hinges on the fact that a pair (A, P) is a max pair of Q iff (A', P') is a max pair of Q'. Furthermore, then A conches x in Q iff A' conches x' in Q'. Then, $P = \sqrt{x^{-1}A}$ iff $P' = \sqrt{x'^{-1}A'}$, etc.

Appendix B

The Conch Intersection Theorem for rings with few zero divisors

The proof of the second theorem in Section 8 also suffices to prove:

Theorem. *If R has few zero divisors, then the Conch Intersection Theorem holds for $Q = Q_c(R)$.*

This follows since if S is a subring of Q properly containing R, then for every $z \in S \setminus R$ there corresponds $u \in R$ so that $w = z + u \in Q^*$. Thus, w is a unit of Q lying in S but not in R, sc the theorem follows.

The theorem is a sharpening of Griffin's theorem in Section 7 since a conch subring is a valuation ring for Q, and it also provides an alternative proof.

Converse to the Local FPF Theorem

We note a converse to the local FPF Theorem of Section 12.

Theorem. *If A is a local valuation ring for $Q_c(A)$, and if $Q_c(A)$ is self-injective, then A is FPF.*

By using the equivalence $P_1 \Leftrightarrow P_4$ in G.4 of Section 7, we see that A is an I-ring inasmuch as $A = A_{[M]}$ is integrally closed, where M is the unique maximal ideal, so A is FPF by the second corollary in Section 8.

Local valuation rings of chain rings are chain rings

Every known valuation ring of a chain ring is a chain ring. The next theorem characterizes this property.

Theorem. *For a valuation ring A of a chain ring Q the following are equivalent:*

(1) *A is a local ring.*

(2) *A is a sandwich subring of Q.*

(3) *A is a chain ring.*

Proof. (1)$\Rightarrow$(2) by the Local Valuation Ring Theorem in Section 13, and (2)$\Rightarrow$(1) by the Partial Converse in the same section. Moreover, (3)$\Rightarrow$(1) is trivial. Next assume (1), equivalently (2). If $a, b \in A$, then $aQ \subseteq bQ$ or $bQ \subseteq aQ$. Suppose that $aQ \subseteq bQ$, and write $a = bq$ for some $q \in Q$. If $q \notin A$, then $q \notin J$, hence $q^{-1} \in Q$, so $q^{-1} \in A$. Then $b = aq^{-1}$, hence $bA \subseteq aA$. On the other hand, if $q \in A$, then $aA \subseteq bA$, hence A is chained.

Problems

(1) Does the FPF Conch Intersection Theorem hold for an arbitrary self-injective ring Q? If J is the Jacobson radical of Q, then $\bar{Q} = Q/J$ is a self-injective von Neumann regular ring, so the FPF Conch Intersection Theorem does hold for $\bar{Q}$. It easily follows that $R + J$ is the intersection of FPF conch subrings of Q if $\bar{R}$ is integrally closed in $\bar{Q}$. If, for example, J is nil, then integral closure of R implies that $R \supseteq J$ so $\bar{R}$ is integrally closed in $\bar{Q}$, and hence R is an FPF conch intersection in this case. Does this hold in general?

(2) Is a conch subring on valuation ring A of a ring Q necessarily an order in $Q_c(Q)$? The answer is "yes" if Q is generated over A by units, hence if Q is a local ring, or when Q is regular.

(3) If A and B both conch x in Q, how are they related? For example, suppose Q is a field, and A and B are maximal x-conch subrings. If A is an equi-characteristic complete discrete valuation ring, then we know that $A \approx A/P\langle t \rangle$, the power series ring over A/P. If B is also a complete DVR, then $A \approx B$ iff $A/P \approx B/P$. A dubious conjecture: $A \approx B$ in general.

(4) Which continuous regular rings are integrally closed in their maximal quotient rings?

(5) What are the relative weak global dimensions of conch subrings of rings?

(6) We propose to study conch subrings of non-commutative rings. All of the foregoing theorems on conch rings, except for the case Q is a field, were obtained ring-theoretically, that is, without employing valuation theory. It seems likely that such is the case here, and that ring theory can make a contribution to the structure of intractable rings, e.g. the enveloping algebra of a Lie algebra is an Ore domain, but little else is known about its structure. J. Towber has suggested (orally) that conch rings may be useful here.

(a) A special case occurs when A conches an element x in Q, where x belongs to the center C of Q. If Q is a skew-field, then $A\cap C$ is a chain ring; if Q is a regular ring, then $A\cap C$ is a semihereditary ring. When Q is regular and self-injective, then $A\cap C$ is FPF.

These may be useful for the structure of A.

When Q is a skew-field, I conjecture that A is a chain ring.

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